Topic questions

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Contents

1 Introductory Notes

While every effort has been made to fix errors and typos, some may still remain. Please do get in touch to let me know should you encouter any.

These questions should require only A-level content. They may however introduce new, slightly off-spec ideas. If any off-spec results are required to solve the question, they should be given at the start of the question.

The questions vary *significantly* in difficulty, so please bear this in mind. In general also please do not read the solutions immediately as that would ruin the question!

The red box at the start of each question details the content required in the question. The green box lists any results, or ideas, that you may like to use to answer the question - anything listed in these boxes is assumed without proof.

A value of an argument x a function $f(x)$ (where x is not necessarily a number) is said to be a *zero* or a *root* of the function f if $f(x) = 0$ I will call this a *root* in this text.

In A-level, improper integrals <u>must</u> have the limits considered properly. (For example, if a limit of an integral is ∞, you would need to write $\lim_{a\to\infty} a$.) I will not write that in this text but please do bear in mind.

For some questions with vectors, I have written column vectors as the transpose of row vectors. This is solely in the interest of formatting. (So, $(x, y)^{\mathrm{T}}$ is to mean $\binom{x}{y}$.)

Issues of convergence need not be considered unless explicitly stated in a question.

Some questions may reference *generating functions* (some don't but still use the ideas subtly). These are off-spec for A-level but all you need to know is as follows: if you have a sequence of numbers $(a_n)_n$, then the generating function of this sequence is $G(x) = \sum_{k=0}^{\infty} a_k x^k$.

At the start of the stats question, I have listed a few things which are not on A-level but are needed in order to answer the questions. I have written these at the start of the section in order to avoid repeating myself consistently later on.

Order of summation and integration: these can be interchanged as long as the function being summed or integrated (or both) is sufficiently 'well-behaved'. In practice this relates to things like convergence and continuity. Unless explicitely stated, the order of any sums or integrals may be interchanged here.

2 Questions

2.1 Core Pure

1.

Prerequisites:

Integration by parts, substitution.

Results assumed for this question :

You are given that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

You are also given that for a positive integer n, $n!!$ is the product of all positive integers up to and including n that have the same parity as n. So, for example, $4!! = 4 \cdot 2$ and $3!! = 3 \cdot 1$.

You are given that $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for any fixed $n > 0$.

(a) For fixed real values μ, σ, find the value of

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}\,dx.$$

(b) Let

$$\Gamma(n) = \int_0^{\infty} x^{n-1}e^{-x}\,dx.$$

Show that, for $n > 0$, $\Gamma(n+1) = n\Gamma(n)$. Show that $\Gamma(\frac{k}{2}) = \frac{(k-2)!!}{2^{\frac{k-1}{2}}}\Gamma(\frac{1}{2})$, for k an odd positive integer. Hence find an expression for $\Gamma(\frac{k}{2})$, again for k an odd positive integer.

2.

Prerequisites:

Euler form of a complex number (not necessary).

(a) Show that for any complex numbers a, b, $\overline{a+b} = \bar{a} + \bar{b}$ and $\overline{ab} = \bar{a}\bar{b}$. Hence show that if a complex number m is a root of the equation $\mathcal{P}_n(m)$, where $\mathcal{P}_n(m)$ is an nth degree polynomial in m with real coefficients, then $\bar{m}$ (m conjugate) is another root.

(b) Show also that if for every root m of a polynomial, $\bar{m}$ is another root, and if this polynomial is monic (leading coefficient of 1), then the polynomial has all real coefficients.

(c) Deduce that any polynomial of odd degree has at least one real root. (You may assume that any polynomial of degree n has precisely n, possibly repeated, zeros.)

3. **Prerequisites:**

You are given that a sequence $(x_n)_n$ of real numbers converges if the sequence is monotonic (strictly increasing or decreasing) and bounded (there exist real numbers a, b such that $a < x_n < b$ for all n.

Prove by induction that

$$\sum_{r=1}^{n} \frac{1}{r^2} < 2$$

for $n \geq 1$. Prove that this series is convergent. (You might find it useful to prove a stronger result, from which the desired inequality would follow.)

4. **Prerequisites:**

Cross-product.

(a) Show that the cross-product is non-associative (that is, $a \times (b \times c) = (a \times b) \times c$ does not necessarily hold).

(b) Prove that there does not exist a vector $e \in \mathbb{R}^3$ such that for any $x \in \mathbb{R}^3$, $e \times x = x$. Why would it not make sense for there to exist such a vector?

5. **Prerequisites:**

Roots of polynomials.

Given the numbers $a_1, a_2, \ldots, a_n$ and a polynomial $P(z)$ of degree n of which each of the $a_i, i \in \{1, \ldots, n\}$

are roots, show that you can find an expression for $a_1^n + a_2^n + \cdots + a_n^n$ without explicitly determining any of the a_i.

Prerequisites:

6. Proof by induction

(a) For a prime p, explain why or show that $\binom{p}{r}$ is divisible by p for $0 < r < p$.

(b) Prove by induction on a that

$$a^p \equiv a \pmod{p}$$

for any natural number a.

Prerequisites:

7. Proof by induction

(a) Prove by induction that

$$\frac{d^n}{dt^n}[f(t)g(t)] = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(t)g^{(k)}(t)$$

for $n \geq 1$. You may assume all the derivatives are defined.

(b) Let $f^{(n)}(t) = e^{at}$ for some constant a. By considering part (a) and choosing a suitable function $g(t)$, show that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

for any fixed constants.

Prerequisites:

8. Implicit differentiation

Results assumed for this question :

You are given than the equation of an arbitrary conic is given by

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

for constants a, b, c, d, e, f.

(a) Show that the tangent to a conic at a point (x_0, y_0) can be found by replacing x^2 with xx_0, y^2 with yy_0, xy with $\frac{xx_0+yy_o}{2}$ and linear x, y terms with $\frac{x+x_0}{2}$, $\frac{y+y_0}{2}$ respectively.

(b) Find the point of intersection of the tangents to the unit circle at the points $(1, 0)$ and $(0, 1)$.

9.

Prerequisites:

Coordinate geometry

(a) Given a point P and a line L, show that the shortest line segment joining them is perpendicular to the line L.

(b) Find the locus of points equidistant from the point $(p, 0)$ and the line $x = -p$.

10.

Prerequisites:

First order differential equations.

Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}^+$ satisfying

$$f(x) = \int_{-\infty}^{x} f(x)\, dx$$

with $f(0) = 1$.

11.

Prerequisites:

Intermediate Value Theorem.

By considering $\int_0^1 x\, dx$, or otherwise, show that if a continuous function f satisfies

$$\int_0^1 f(x)\, dx = \frac{1}{2},$$

then there exists $x_0 \in (0, 1)$ such that $f(x_0) = x_0$.

12.

Prerequisites:

Dot product of vectors

(a) Consider the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}, \mathbf{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$. Show that $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x}||\mathbf{y}|$. Hence show that for any $2n$ real numbers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ the following inequality holds:

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2.$$

When does equality hold?

(b) Consider once again the vector inequality $\mathbf{x} \cdot \mathbf{y} \leq |\mathbf{x}||\mathbf{y}|$. Use this inequality to show that, for any vectors $\mathbf{x}, \mathbf{y}$, the inequality

$$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$$

holds. When does equality hold?

13.

Prerequisites:

Cross-product of vectors.

Show (without the use of a calculator) that the area enclosed by the parallelogram formed by the vectors

$\mathbf{a} = (1, 2, 3)^{\mathrm{T}}$ and $\mathbf{b} = (2, 3, 4)^{\mathrm{T}}$ is equal to the modulus of their cross-product.

14.

Prerequisites:

Complex number geometry.

Results assumed for this question :

You are given the *triangle inequality*. That is, for any complex numbers a, b, $|a + b| \leq |a| + |b|$ and that $|a + b| = |a| + |b|$ if and only if there exist $l, k \in \mathbb{R}^+$ such that $lb = ka$.

You are also given that four points A, B, C, D (in that order) (non-colinear) form a cyclic quadrilateral if and only if the opposite angles sum to 180°(you may already know this from GCSE).

(a) By considering the *triangle inequality*, show that, for any complex numbers a, b, c, d, the inequality

$$|a - b||c - d| + |a - d||b - c| \geq |a - c||b - d|$$

holds.

(b) Consider 3 non-collinear points A, B, C represented by complex numbers a, b, c respectively. By considering $\arg \frac{m}{n} = \arg m - \arg n$ for any complex numbers m, n, find $\angle ABC$ as the argument of a single complex number (you need not simplify your expression).

(c) Show that equality in part (a) holds if and only if the numbers a, b, c, d represent the vertices of a cyclic quadrilateral.

15.

Prerequisites:

Series expansions.

Show that the Maclaurin Series of a polynomial is the polynomial itself.

16.

Prerequisites:

The ideas behind a proof by induction.

In this question, all of the $a_i, i \in \mathbb{N}$ are non-negative real numbers.

(a) Show that for any a_1, a_2, the inequality

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$$

holds. When are they equal?

(b) By considering part (a), prove that, if

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n},$$

then

$$\frac{a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots + a_{2n}}{2n} \geq \sqrt[2n]{a_1 a_2 \ldots a_{2n}}.$$

(c) Consider the expression

$$\sqrt[n+1]{a_1 a_2 \ldots a_n a_{n+1}}.$$

Find a possible choice for a_{n+1} in terms of $a_1, a_2, \ldots a_n$ such that the given expression reduces to

$$\sqrt[n]{a_1 a_2 \ldots a_n}.$$

(d) Show that, if

$$\frac{a_1 + a_2 + \dots a_n + a_{n+1}}{n+1} \geq \sqrt[n+1]{a_1 a_2 \dots a_n a_{n+1}},$$

then

$$\frac{a_1 + a_2 + \dots a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n}.$$

(e) Do these steps demonstrate that for any $n \in \mathbb{N}, n \geq 2$, the inequality

$$\frac{a_1 + a_2 + \dots a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n}.$$

holds?

17.

Prerequisites:

Integration by Parts, the Squeezing Principle

Let

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n(x)\, dx,$$

where n is a non-negative integer.

(a) Find an expression for I_n in terms of I_{n-2}.

(b) Find an expression for I_n, distinguishing between the cases that arise.

(c) By noting that $\sin^{2k+1} x < \sin^{2k} x < \sin^{2k-1} x$ for all $x \in (0, \frac{\pi}{2})$, or otherwise, show that

$$\left[\frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)}\right]^2 \cdot \frac{2}{2n+1} < \pi < \left[\frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)}\right]^2 \frac{2}{2n}.$$

(d) Find

$$\lim_{n \to \infty} \frac{1}{n} \left[\frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)}\right]^2.$$

18.

Prerequisites:

Factor theorem.

Results assumed for this question :

For this question, assume the Factor Theorem holds for power series exactly as it does for polynomials. That is, if all the roots (call them x_i) of a power series are known, the the power series can be expressed as some constant multiplied by $\prod_i (x - x_i)$.

(a) By considering the small angle approximation for $\sin x$, find $\lim_{x\to 0} \frac{\sin x}{x}$.

(b) By considering the 'Factor Theorem', show that

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{1^2\pi^2}\right) \cdot \left(1 - \frac{x^2}{2^2\pi^2}\right) \cdot \left(1 - \frac{x^2}{3^2\pi^2}\right) \cdots .$$

(c) Find

$$\frac{2}{1} \cdot \frac{2}{3} \cdots \frac{4}{3} \cdot \frac{4}{3} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdots ,$$

given this product converges to a finite, non-zero value.

19.

Prerequisites:

Second-order linear differential equations whose auxiliary equations have complex solutions.

First order recurrence relations.

This question concerns the differential equation

$$y''(x) + 4y(x) = 2$$

with initial conditions $y(0) = y'(0) = 0$.

(a) Solve this differential equation.

(b) Suppose this differential equation has a solution of the form $y(x) = \sum_{k=0}^{\infty} a_k x^k$. By considering the initial conditions, find the values of a_0 and a_1. By considering the coefficient of x^k, show that, for $k \geq 1$, the following recurrence relation holds:

$$(k + 2)(k + 1)a_{k+2} + 4a_k = 0.$$

Hence show that $a_{2k} = \frac{2 \cdot 4^{k-1}(-1)^{k-1}}{(2k)!}$. Deduce the series expansion for $\sin^2 x$.

(c) By using standard series (those given in the formula booklet), verify this series.

20.

Prerequisites:

Trigonometric integration.

Results assumed for this question :

You are given that

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

In this question, all of the b_i and L are real constants. Also, m and n are positive integers.

(a) Evaluate the integral

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx,$$

distinguishing between the cases that arise.

(b) Let $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{m\pi x}{L}\right)$. What is the period of this function? Show that

$$\frac{2}{L}\int_0^L [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2$$

(c) You are given that for $f(x) = x, x \in [0,2), f(x+2) = f(x)$, $\quad f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}\right)$.
Find the value of

$$\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

21.

Prerequisites:

2×2 matrices

Here, $\mathbf{O}_2$ denotes the 2×2 zero matrix.

Consider a 2×2 matrix $\mathbf{M}$ given by

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

, where all the entries are non-zero.

Is it true that there exist non-zero matrices $\mathbf{A}, \mathbf{B}$ such that $\mathbf{MA} = \mathbf{BM} = \mathbf{O}_2$ if and only if $\mathbf{M}$ is singular?

22.

Prerequisites:

Volumes of revolution.

Results assumed for this question :

You are given that the formula for the surface area of a function $f(x)$ defined on (a, b) and rotated about the x-axis by 2π radians is

$$S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\ dx$$

Gabriels's horn is the solid formed by rotating the curve

$$y = \frac{1}{x}, \quad x \in [1, \infty)$$

around the x-axis by 2π radians.

(a) Show that the volume of Gabriel's horn is π.

(b) By considering

$$I = \int_1^\infty \frac{1}{x} dx$$

or otherwise, show that Gabriel's horn has infinite surface area.

23.

Prerequisites:

First and second order differential equations.

Results assumed for this question : I

n this question it may be used without proof that the derivative of an integral with respect to a different parameter may be 'allowed inside the integral'. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_a^b f(x,t)\,\mathrm{d}x\right] = \int_a^b \frac{\partial f(x,t)}{\partial t}\,\mathrm{d}x.$$

(a) Let f be the following function

$$f(\alpha) = \int_0^\infty \frac{\sin x}{x} e^{-\alpha x}\,\mathrm{d}x$$

where α is a non-negative real number.

i. Show that

$$f'(\alpha) + \frac{1}{1+\alpha^2} = 0.$$

ii. Hence, show that

$$\int_{-\infty}^\infty \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$

(b) Let g be the function

$$g(\alpha) = \int_0^\infty \frac{\cos \alpha x}{x^2+1}\,\mathrm{d}x$$

where α is a non-negative real number.

i. Show that

$$g'(\alpha) + \frac{\pi}{2} = \int_0^\infty \frac{\sin \alpha x}{x(x^2+1)}\,\mathrm{d}x$$

and hence that $g''(\alpha) = g(\alpha)$.

ii. Show that

$$\int_{-\infty}^\infty \frac{\cos x}{x^2+1}\,\mathrm{d}x = \frac{\pi}{e}.$$

Prerequisites:

24. Hyperbolic integration.

By considering the integral

$$\int_0^x \frac{1}{1-t^2}\,dt$$

in two ways, find an expression for $\operatorname{artanh}(x)$ in terms of a logarithmic expression in x.

25.

Prerequisites:

Graph sketching, polar coordinates.

(a) Sketch the curve $y = \sqrt{\cos x + \sin x}$ for $x \in [-\pi, \pi]$. Find the coordinates of the maximal value of this curve.

(b) Sketch the curve $r = \sqrt{\cos \theta + \sin \theta}$ (polar coordinates here), over an appropriate interval.

(c) Find the area enclosed by the curve in part (b).

(d) Find the points where the tangent to the curve in part (b) is a vertical line.

26.

Prerequisites:

Hyperbolic integration.

Let

$$I(x) = \int_0^x \tanh(t)\, dt$$

where $x \geq 0$.

I believe that $I(x) = \ln \sinh x$.

(a) Sketch the graph $y = \tanh x$.

(b) By considering $\lim_{x \to 0} I(x)$, explain why I am wrong. Find the correct function.

(c) For which values of x will we have the same value of $I(x)$?

27.

Prerequisites:

Complex loci.

(a) Describe using words the locus of complex numbers z satisfying $|z-a| = |z-b|$ for fixed complex numbers a, b.

(b) Sketch the loci of z in each of the following:

i. $|z-(1+i)| = |z-2|$.

ii. $|z-(1+i)| \leq |z-2|$.

iii. $|z-(1+i)| < |z-2|$

(c) i. Consider the normal xy-plane. Find the locus of such that the distance to the origin is twice the distance to the point $(0, 1)$. Sketch this locus. What shape is this?

ii. Back to complex numbers. Sketch the locus of points z satisfying $|z-(1+i)| \leq 2|z-2|$ (a rough sketch would suffice).

28.

Prerequisites:

Euler form of complex numbers.

Consider the nth order polynomial $T_n(x)$ defined recursively by $T_0(x) = 1, T_1(x) = x, T_{n+1} = 2xT_n(x) - T_{n-1}(x)$.

(a) Prove that $T_n(\cos(\theta)) = \cos(n\theta)$. Hence find the maximum value of $T_n(x)$ on the interval $[-1, 1]$.

(b) Explain why $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$.

(c) Prove that

$$T_n(x) = \frac{1}{2}\left[\left(x + i\sqrt{1-x^2}\right)^2 + \left(x - i\sqrt{1-x^2}\right)^2\right]$$

2.2 Stats

Notes :

Throughout the entirity of this section, $q = 1 - p$.

The probability density function of the normal distribution with mean μ and variance σ^2 is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

for all real x.

The covariance of two random variables X, Y is defined by $\text{Cov}(X, Y) = \mathbb{E}((X - \mu_x)(Y - \mu_y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$, where $\mu_x = \mathbb{E}(X), \mu_y = \mathbb{E}(Y)$.

It may be helpful to note in this section that

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = 1$$

called the Gaussian integral.

1.

Prerequisites:

Expectation Algebra

Results assumed for this question :

For this question, you are given that $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

(a) Find an expression for $\text{Var}(X + Y)$ in terms of $\text{Var}(X), \text{Var}(Y)$ and $\text{Cov}(X, Y)$.

(b) Find a necessary and sufficient condition on $\text{Cov}(X, Y)$ for $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ to hold.

(c) Demonstrate that if X and Y are independent then this condition holds. (You should show this from the definition of independence - that $\mathbb{P}(X \cap Y) = \mathbb{P}(X)\mathbb{P}(Y)$.)

(d) Let $Z = aX + Y$, for some real constant a. By considering a quadratic in a, show that $(\text{Cov}(X, Y))^2 \leq \text{Var}(X)\text{Var}(Y)$.

2.

Prerequisites:

Binomial distribution.

(a) By considering a binomial distribution as a sum of outcomes independent Bernoulli Trials, find the mean and variance of a Binomial Distribution with probability of success p and number of trials n.

(b) By considering the derivative of the function $f(x) = (px+q)^n$ in two ways, find the expectation of a binomial distribution with probability of success p and number of trails n.

(c) By considering the series

$$\sum_{k=0}^{n} k^2 \binom{n}{k} p^k q^{n-k} x^k$$

for a suitable value of x, show that the variance of a Binomial Distribution is npq.

3. **Prerequisites:**

Binomial distribution.

(a) Consider a pile of n red beads and m blue beads. Write down, in terms of a single binomial coefficient, the number of ways of choosing k beads from this pile. Consider instead picking r beads from the red pile and $k-r$ from the blue pile, where $r \geq 0$. In how many ways can this be done? Write your answer in terms of two binomial coefficients. Take the sum over all values of r between 0 and k (leave in summation form). What does this sum represent?

(b) Given two independent random variables $X \sim B(n,p)$ and $Y \sim B(m,p)$ and $Z = X+Y$, by considering $\mathbb{P}(Z=z)$, find the distribution of Z in terms of m, n and p.

4. For this question you may assume that $\ln 2 < 0.7$. CHANGE THIS QUESTION CONSIDERABLY

(a) You play a game in which you throw a fair coin until the coin lands on heads. State the distribution and the probability of getting heads on the kth throw.

In this game you win £1 if k is odd and lose £1 if k is even. Find your expected winnings.

(b) This time, consider the same game but with a different pay-off. If k is odd, you win £$\frac{2^k}{k}$, if k is even then you win the same amount. Show that your expected winnings can be written as

$$\sum_{k=1}^{\infty} \left((-1)^{k-1} \frac{2^k}{k} \right) \frac{1}{2^k}.$$

By considering the series expansion of $\ln(1+x)$, find a possible value of this sum.

Now, consider rearranging the elements of the sum. Show that they may be rearranged to give

$$\sum_{k=1}^{\infty} \left(\frac{1}{2k-1} + \frac{1}{2k+1} - \frac{1}{2k} \right).$$

Deduce that the terms can be rearranged so that the the value of the series exceeds $\frac{5}{6}$ and hence that the series can be shown to approach two distinct values.

5.

Prerequisites:

Matrix diagonlisation.

Consider the 2×2 matrix:

$$\mathrm{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with entries a, b, c, d given positive, real numbers. I wish to model the weather using this matrix. In particular, given the probability of sun (x) and probability of rain (y) on day k as a vector $\binom{x}{y}$, I wish to find the probability of sun and rain on day $k+1$. For this question, the only two (mutually exclusive) categories of weather are sun and rain.

(a) What is the value of $x + y$? Which of a, b, c, d represents the probability of sun on day k followed by sun on day $k+1$? Why do we necessarily have $(a + c) = (b + d) = 1$?

(b) From observation I find the following: the probability of sun on day 1 is $\frac{1}{2}$, $a = \frac{9}{10}$ and $d = \frac{1}{5}$. What is the probability of sun on a day far in the future?

6.

Prerequisites:

Binomial distribution, Poisson distribution.

Consider buses A and B arriving at my local bus stop at the uniform average rate of μ buses per hour

and λ buses per hour respectively. Given that I saw $n \in \mathbb{Z}^+$ buses in an hour of observing, what is the probability that I saw x buses of type A in that hour?

7.

Prerequisites:

Integration.

Results assumed for this question :

The *Moment Generating Function* of a random variable X with distribution function F_X is defined as $M_X(t) = \mathbb{E}(e^{tX})$.

You are given that a *Moment Generating Function* uniquely determines a distribution (that is, if random variables X and Y have the same *Moment Generating Function*, then $\mathbb{P}(X = t) = \mathbb{P}(Y = t) \quad \forall t$).

(a) Find the *moment generating function* of a random variable normally distributed with mean μ and variance σ^2.

(b) You are given that, for independent random variables M, N, $\quad \mathbb{E}(MN) = \mathbb{E}(M)\mathbb{E}(N)$.

Find, for X and Y independent random variables with *moment generating functions* $M_X(t), M_Y(t)$ respectively, the moment generating function of $X + Y$.

(c) Let $X \sim N(\mu_x, \sigma_x^2), Y \sim N(\mu_y, \sigma_y^2)$ and let $Z = X + Y$. Show that $Z \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$.

8.

Prerequisites:

General expectation algebra.

Results assumed for this question :

You are given that $\text{Cov}^2(X, Y) \leq \text{Var}(X)\text{Var}(Y)$, with equality holding if $Y = aX + b, a \neq 0$.

The PMCC $\rho_{X,Y}$ is defined as $\frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$ where σ_x, σ_y are the standard deviations of X, Y respectively.

(a) Find an upper and lower bound on $\rho_{X,Y}$, giving an example of when each of these bounds may be achieved.

(b) hmmm lost once again.

Note: SRCC from PMCC and maximing sum of over all permutations.

9.

Prerequisites:

Poisson distribution.

Given $X \sim Po(\lambda)$, show that $\mathbb{E}(X) = \text{Var}(X) = \lambda$.

10. **Prerequisites:**

Continuous probability distributions (off-spec for A-level).

Consider the curve $y = x^2$ and the line $y = mx$, where $m \sim U(0, 1)$. Find the expected finite area enclosed by these curves.

11. **Prerequisites:**

Hypothesis testing, Poisson distribution.

Results assumed for this question :

You are given that $\frac{60}{e^6} > 0.14$.

My local train station has one platform and is served by trains A, B, C, each of which are said arrive at a uniform average rate of once per hour, twice per hour and three times per hour respectively. I observe my station for one hour and find that only 3 trains arrive in that time. Test, at the 10% significance level, whether or not the trains do indeed arrive at the given rate.

12. **Prerequisites:**

Geometric distribution.

(a) Consider the series

$$\sum_{k=1}^{\infty} kp^{k-1}qx^{k-1}.$$

By considering this as the derivative of some other series, find the expectation of a geometric distribution.

(b) In a similar way, find the variance of a geometric distribution.

13. **Prerequisites:**

General inequalities, expectation algebra.

(a) For a discrete random variable X, prove that

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

(b) Prove that

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2},$$

where $\mu = \mathbb{E}(X)$ and $\sigma^2 = \mathrm{Var}(X)$.

14.

Prerequisites:

Definition of variance.

Prove that the variance of a random variable is non-negative. When is it equal to zero?

2.3 Mechanics

1. Consider an object of mass m moving in one line and subject to one (not necessarily constant) force F acting on this object. The force F acts on this object from point a to point b, at which points the objects velocities are v_a, v_b respectively. Demonstrate that the Work-Energy principle holds in this scenario.

2. Try to get the stacking books on top of each other (harmonic series) question here.

2.4 Extra Pure

1. **Prerequisites:**

   Group theory, Lagrange's Theorem, modular arithmetic.

   Results assumed for this question :

   For this question you may assume that the set of integers $\{1, 2, \ldots, p-1\}$ have a multiplicative inverse modulo p.

   (a) Prove that the numbers $\{1, 2, \ldots, p-1\}$ form a group under multiplication modulo p for any prime number p. Call this group G.

   (b) Explain why the order of an element in a group of order p (p is finite) must divide p.

   (c) Fix an element $a \in G$ with order k. By noting that $(p-1) = mk$ for some integer m, show that

   $$a^{p-1} \equiv 1 \pmod{p}.$$

2. **Prerequisites:**

   Group theory.

   Results assumed for this question :

   You are given that for n prime, the set of integers $\{1, 2, \ldots, n-1\}$ with the binary operation of multiplication modulo n form a group if n is prime.

   Consider the set of integers $\{1, 2, \ldots, n-1\}$ modulo n. You are given that, for n prime.

   (a) Prove that if n is composite, then these numbers do not form a group.

   (b) Prove that the equivalence

   $$(n-1)! \equiv (-1) \pmod{n}$$

   holds if and only if n is prime.

3. **Prerequisites:**

   Group theory.

   Given a natural number n, let $\varphi(n)$ denote the number of positive integers less than or equal to n and coprime to n. By considering a relevant group with the binary operation of multiplication, or otherwise, show that, for any positive integer a,

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

Deduce that, for p prime,

$$a^p \equiv a \pmod{p}$$

for any positive integer a.

4.

Prerequisites:

General algebra.

Note: The parts of this question build up to proving a final result in part (c). You might like to complete part (c) only for an extra challenge.

A *Pythagorean triple* is a set of 3 positive integers (a, b, c) such that $a^2 + b^2 = c^2$. This set is called *primitive* if $\gcd(a, b, c) = 1$.

A triangle with integer side lengths will be called *integer* in this question. Consider a triangle A with side lengths $a, a + r, a + 2r$ where a and r are positive.

(a) Find a necessary and sufficient condition on a and r for A to be a non-degenerate triangle.

For the remainder of this question, A is a right-angled triangle.

(b) Show that a and r satisfy the equation

$$a^2 = 2ar + 3r^2$$

and hence show that $a = 3r$.

(c) Hence show that the only right-angled *integer* triangle with its sides in an arithmetic progression has side-lengths $(3, 4, 5)$.

5.

Prerequisites:

General algebra.

Note: The parts of this question build up to proving a final result in part (d). You might like to complete part (d) only for an extra challenge.

A triangle with integer side lengths will be called *integer* in this question. This question is about a triangle A with side-lengths (a, ar, ar^2) in this question, with $a > 0, r > 1$. Throughout this question, triangle A is right-angled.

(a) Explain why, for A to be *integer*, $a \in \mathbb{Z}$ and $r \in \mathbb{Q}$ (but not necessarily $r \in \mathbb{Z}$).

(b) Show that r must satisfy
$$1 + r^2 = r^4.$$

(c) Consider the equation in r given in the previous part. Suppose there is a rational solution $r = \frac{p}{q}, \gcd(p, q) = 1$ to the equation. By multiplying both sides of the equation by q^3 show that q must divide p^4 and deduce the value of q. Deduce that if a solution to the equation is rational, then it is also integer.

(d) Show that there are no right-angled *integer* triangles.

6.

Prerequisites:

Matrix diagonalisation.

Consider the sequence defined by $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}, n \geq 1$.

(a) Let $\mathbf{f}_n = \binom{F_n}{F_{n-1}}, n \geq 1$. There is a matrix $\mathbf{A}$ such that $\mathbf{A}\mathbf{f}_n = \mathbf{f}_{n+1}$. Show that
$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(b) Show that $\mathbf{A}$ can be written as INSERT.

(c) Hence find a closed form for F_n.

(d) Suppose
$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$
exists. Find this limit.

7.

Prerequisites:

Matrix diagonalisation.

Let $\mathbf{A}$ be an $n \times n$ diagonalizable matrix with $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where $\mathbf{D}$ is diagonal (with real entries) and P is invertible.

(a) Find (*not* write down) an expression for $\mathbf{A}^k$ in terms of $\mathbf{P}, \mathbf{D}, \mathbf{P}^{-1}$ and k, where $k \in \mathbb{Z}, k \geq 0$.

(b) We now seek a matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$. Note that we can write this as $\sqrt{\mathbf{A}} = \mathbf{B}$, if we take the principle roots of the entries in the diagonal matrix of $\mathbf{A}$. Using the diagonalisation of $\mathbf{A}$, propose an expression for $\mathbf{B}$ in terms of $\mathbf{P}, \mathbf{P}^{-1}$ and $\mathbf{D}$. Verify your claim.

(c) Under what conditions on the eigenvalues of $\mathbf{A}$ will $\mathbf{B}$ be a real matrix?

(d) Generalize this idea: how would you compute $\mathbf{A}^{\frac{p}{q}}$ for integers p, q ($q > 0$)? Find, with explanation, any necessary restrictions on the eigenvalues for $\mathbf{A}^{\frac{p}{q}}$ to be a real matrix, distinguishing between the cases that may arise.

(e) Finally, how would you define A^r for an arbitrary real number r? What must be true about the eigenvalues of A in this case? What happens if the matrix is non-diagonalisable?

8.

Prerequisites:

Multivariable calculus.

Results assumed for this question :

For this question, you are given that if a function $f(x, y, z)$ is subject to a constraint $g(x, y, z) = C$ (C constant) and has a maximal or minimal value, then this maximal or minimal value occurs at a point (x, y, z) at which the gradients of $f(x, y, z)$ and $g(x, y, z)$ are parallel (that is, $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$).

This question is about finding the minimal value of the function

$$f(a, b, c) = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

over all positive real values a, b, c.

(a) Explain why f has a minimal value but does not have a maximal value. Suppose for some values of a, b, c such that $a + b + c = t$, $f(a, b, c) = x$. Show that if we have $a = ta_1, b = tb_1, c = tc_1$, then $f(a_1, b_1, c_1) = x$. Explain why this means that the minimal value attained by f over all real positive values is also attained for some values a, b, c such that $a + b + c = 1$.

<u>From here, you may assume that</u>

$$f(a, b, c) = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

<u>and that</u> $a + b + c = 1$.

(b) Show that

$$f(a,b,c) = \frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c}$$

and explain briefly why $0 < a, b, c < 1$.

(c) Show that, at the minimal value of $f(a,b,c)$, $a = b$ and explain why it follows that $a = b = c$. What is the minimal value of $f(a,b,c)$?

9.

Prerequisites:

Group theory.

For an element $a \in G$ where G is a group, consider the set $N(a) = \{x \in G | ax = xa\}$. Prove that $N(a)$ is a subgroup of G.

10.

Prerequisites:

Group theory.

Results assumed for this question :

You may assume that the real numbers are closed under addition and multiplication, and that these binary operations are associative on the real numbers.

The imaginary unit i is defined as a solution to the equation $x^2 + 1 = 0$. This question is about investigating the possibility of defining this complex unit in a more general way.

(a) Prove that any quadratic equation of the form $x^2 + px + q = 0$ can be solved in complex numbers.

(b) Suppose instead we define a complex unit I as a solution to the equation $x^2 + px + q = 0$, call these *generalised* complex numbers. Find an expression for I in terms of i.

(c) Prove that complex numbers form a group under the binary operation of addition, and under multiplication with the value 0 removed.

(d) You are given that *generalised* complex numbers also form a group under addition and under muliplication with the value 0 removed. By considering the expression found in part (b), or otherwise, prove that these groups are isomorphic to each other.

3 Solutions

3.1 Core Pure

1. Consider the substitution $u = \frac{x-\mu}{\sqrt{2}\sigma}$. The limits are changed from $(-\infty, \infty)$ to $(-\infty, \infty)$ and $\sqrt{2}\sigma\, du = dx$. So the integral becomes

$$\begin{aligned}&\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2}\sqrt{2}\sigma\, du\\ =&\frac{1}{\sigma\sqrt{2\pi}}\sqrt{2}\sigma\sqrt{\pi}\\ =&1\end{aligned}$$

The integrand is the probability density function of the Normal distribution with mean μ and variance σ^2. This question shows that the second of the three standard probability axioms holds with this function (that is, the probability of at least one elementary event occurring is 1).

We have

$$\begin{aligned}\Gamma(n+1) &= \int_0^{\infty} x^n e^{-x}\, dx\\ &= [-x^n e^{-x}]_0^{\infty} + \int_0^{\infty} nx^{n-1}e^{-x}\, dx\\ &= [0-0] + n\int_0^{\infty} x^{n-1}e^{-x}\, dx\\ &= n\Gamma(n).\end{aligned}$$

So we have:

$$\begin{aligned}\Gamma\left(\frac{k}{2}\right) &= \frac{k-2}{2}\Gamma\left(\frac{k-2}{2}\right)\\ &= \frac{k-2}{2}\frac{k-4}{2}\Gamma\left(\frac{k-4}{2}\right)\\ &= \frac{(k-2)!!}{2^{\frac{k-1}{2}}}\Gamma\left(\frac{1}{2}\right).\end{aligned}$$

What is left is to evaluate $\Gamma(\frac{1}{2})$:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} x^{-\frac{1}{2}}e^{-x}\, dx$$

Using the substitution $u^2 = x$, we have $(0, \infty) \to (0, \infty)$ and $2u\, du = dx$, so we need to evaluate

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} 2uu^{-1}e^{-u^2}\, du = 2\int_0^{\infty} e^{-u^2}\, du.$$

To find the value of $I = \int_0^{\infty} e^{-u^2}\, du$ we can use the result given at the start of the question, along with the fact that the integrand is even to obtain $I = \frac{\sqrt{\pi}}{2}$ and so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. It follows that $\Gamma(\frac{k}{2}) = \frac{\sqrt{\pi}(k-2)!!}{2^{\frac{k-1}{2}}}$.

(2) (a) Let $a = a_1 + a_2 i, b = b_1 + b_2 i$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. The complex conjugate is $a_1 + b_1 - (a_2 + b_2)i = (a_1 - a_2 i) + (b_1 - b_2)i = \bar{a} + \bar{b}$.
Let $a = r_a e^{i\theta_a}, b = r_b e^{i\theta_b}$. Then $\overline{ab} = \overline{r_a r_b e^{i(\theta_a + \theta_b)}} = r_a r_b e^{-i(\theta_a+\theta_b)} = r_a e^{-i\theta_a} r_b e^{-i\theta_b} = \bar{a}\bar{b}$.

We have $\mathcal{P}_n(m) = a_0 + a_1 m + \cdots + a_n m^n = 0$. Taking conjugates of both sides, we have

$$\begin{aligned} \overline{a_0 + a_1 m + \cdots + a_n m^n} &= 0 \\ \bar{a}_0 + \overline{a_1 m} + \cdots + \overline{a_n m^n} &= 0 \\ a_0 + a_1 \bar{m} + \cdots + a_n \bar{m}^n &= 0 \\ \mathcal{P}_n(\bar{m}) &= 0 \end{aligned}$$

as required.

(b) For the other direction, consider any value m such that $\mathcal{P}_n(m) = 0$. This implies that $\mathcal{P}_n(\bar{m}) = 0$ also and so $\mathcal{P}_n(z)$ can be written as $(z-m)(z-\bar{m})\mathcal{Q}_{n-2}(z)$ for some polynomial $\mathcal{Q}$. But $(z-m)(z-\bar{m}) = (z^2 - (m+\bar{m})z + m\bar{m}) = (z^2 - 2\mathrm{Re}(m) + |m|^2)$, where we can see that all the coefficients are real. This argument holds for any non-real root (if the root m is real, then $m = \bar{m}$ and we don't actually have another root. In the case that m is real however, the polynomial factor $(z - m)$ is already real. So $\mathcal{P}_n(z)$ can be written as the product of real polynomials and so itself is real. Note: finding all the zeros of the polynomial determines the polynomial up to a constant factor. So specifying that the leading coefficient is 1 fixes this constant factor and ensures that you cannot multiply through the polynomial by a non-real number to obtain non-real polynomial with roots all complex conjugate pairs.

(c) Note that roots come in complex conjugate pairs. So if we have one non-real root, we necessarily have another. So non-real roots come in pairs. If none of the roots of the polynomial were real, we would have an even number of roots, which is incorrect. So at least one of them must be real.

3. We can prove a slightly stronger claim, which would then be used to get the desired result. The claim we shall prove by induction is

$$\sum_{r=1}^{n} \frac{1}{n^2} < 2 - \frac{1}{n}$$

for $n \geq 2$ (not 1, as equality holds in that case).

For $n = 1$, we verify the original claim directly to find 1 on the LHS and 2 on the RHS, the assertion is clearly true. For $n = 2$ we verify our new claim to find $1 + \frac{1}{4} = \frac{5}{4} < \frac{3}{2} = 2 - \frac{1}{2}$ as per the claim.

For the inductive step, assume the claim is true for $n = k$, that is

$$\sum_{r=1}^{k} \frac{1}{r^2} < 2 - \frac{1}{k}$$

then we want to show that

$$\sum_{r=1}^{k+1} \frac{1}{k^2} < 2 - \frac{1}{k+1}$$

but by hypothesis,

$$\sum_{r=1}^{k+1} \frac{1}{k^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

so we want to show that

$$\begin{aligned} 2 - \frac{1}{k} + \frac{1}{(k+1)^2} &< 2 - \frac{1}{k+1} \\ \iff \frac{1}{(k+1)^2} &< \frac{1}{k} - \frac{1}{k+1} \\ \iff \frac{1}{(k+1)^2} &< \frac{1}{k(k+1)} \\ \iff \frac{1}{(k+1)} &< \frac{1}{k} \end{aligned}$$

which is clear as k is a positive integer. So the inductive step holds, and the claim is true by induction $\forall n \geq 1$, so

$$\sum_{r=1}^{n} \frac{1}{r^2} < 2 - \frac{1}{n} < 2$$

as required. So this series is bounded from above. We are adding positive terms so it is strictly increasing. Also it is bounded below by 0 (for example). So it is convergent as required. Note: the exact value of this series was the question of the Basel Problem.

In the note at the start of the question, it was said that the result about convergence was for sequences, not series. But we can just define our sequence $x_n := \sum_{r=1}^{n} \frac{1}{r^2}$ so the result given can be applied here.

4. (a) Consider the unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$. $\hat{\imath} \times (\hat{\jmath} \times \hat{\jmath}) = \hat{\imath} \times \mathbf{0} = \mathbf{0}$ but $(\hat{\imath} \times \hat{\jmath}) \times \hat{\jmath} = \hat{k} \times \hat{\jmath} = -\hat{\imath} \neq \mathbf{0}$.

   (b) Suppose such a vector e does exist. Then $e \times e = e$. But any vector crossed with itself gives the zero vector. So $e = \mathbf{0}$. However, for any non-zero vector x, $\mathbf{0} \times x = \mathbf{0} \neq x$.

5. Don't have a solution yet

6. (a) $\binom{p}{r} = \frac{p!}{r!(p-r)!}$. As $0 < r < p$, each term in the expansion of $r!$ and in $(p-r)!$ is less than p. And as p is a prime, it follows that the factor p does not appear in the denominator (only in the numerator). As this is a binomial term, it is an integer. And so the expression is divisible by p.

   (b) For the base case, let $a = 1$. The claim amounts to

$$1^p \equiv 1 \bmod p$$

   which is clearly true. Consider the claim true for $a = k$, then for $a = k + 1$ we have

$$(k+1)^p \equiv k^p + \binom{p}{1} k^{p-1} + \cdots + 1^p \pmod{p}$$

And using part (a) we have $\binom{p}{r} \equiv 0 \pmod p$:

$$\equiv k^p + 0 + \dots 1^p \pmod p$$

And by hypothesis $k^p \equiv k \pmod p$:

$$\equiv k + 1 \pmod p$$

which is the desired result for $k+1$. So the claim is true by induction.

7. (a) Base case, $n = 1$: LHS: $f'(t)g(t) + f(t)g'(t)$ RHS: $\binom{1}{0} f'(t)g(t) + \binom{1}{1} f(t)g'(t) = \text{LHS}$.

Assume the claim true for $n = r$, then for $n = r+1$ the desired result is $\sum_{k=0}^{r+1} \binom{r+1}{k} f^{(r+1-k)}(t)g^{(k)}(t)$. But also

$$\begin{aligned}
\frac{d^{r+1}}{dt^{r+1}}[f(t)g(t)] &= \frac{d}{dt}\left(\frac{d^r}{dt^r}[f(t)g(t)]\right) \\
&= \frac{d}{dt}\left(\sum_{k=0}^{r}\binom{r}{k} f^{(r-k)}(t)g^{(k)}(t)\right) \\
&= \sum_{k=0}^{r}\binom{r}{k}\left(f^{(r+1-k)}(t)g^{(k)}(t) + f^{(r-k)}(t)g^{(k+1)}(t)\right).
\end{aligned}$$

Now, this sum might look daunting. One idea is to consider the binomial coefficients and derivatives of $f(t)$ that pair with $g^{(k)}(t)$ and go from there. That would be

$$\begin{aligned}
&g^{(k)}(t)\left(\binom{r}{k} f^{(r+1-k)}(t) + \binom{r}{k-1} f^{(r+1-k)}(t)\right) \\
&= g^{(k)}(t) f^{(r+1-k)}(t)\left(\binom{r}{k} + \binom{r}{k-1}\right) \\
&= g^{(k)}(t) f^{(r+1-k)}(t)\left(\binom{n+1}{k}\right).
\end{aligned}$$

(The last step can be found by expanding each of the binomial coefficients and simplifying them into one.)

However, we cannot adjoin the $r+1$th derivative of $g(t)$ to anything in this way (as k is indexed up to r), so we are left with

$$\begin{aligned}
\frac{d^{r+1}}{dt^{r+1}}[f(t)g(t)] &= \sum_{k=0}^{r}\binom{r+1}{k} f^{(r+1-k)}(t)g^{(k)}(t) + f^{(0)}(t)g^{(r+1)}(t) \\
&= \sum_{k=0}^{r+1}\binom{r+1}{k} f^{(r+1-k)}(t)g^{(k)}(t)
\end{aligned}$$

as required. So the claim is true by induction.

(b) Setting f to the given equation and $g(t) = e^{bt}$, we have on the LHS $\frac{d^n}{dt^n}[e^{(a+b)t}] = (a+b)^n e^{(a+b)t}$, and on the RHS $\sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} e^{at} b^k e^{bt}$. So cancelling the (non-zero) terms e^{at}, e^{bt} yields the desired result.

8. (a) First, differentiate implicity:

$$\begin{aligned} 2ax + 2by\frac{\mathrm{d}y}{\mathrm{d}x} + cy + cx\frac{\mathrm{d}y}{\mathrm{d}x} + d + e\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x}(2by + cx + e) &= -(2ax + cy + d) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-(2ax + cy + d)}{(2by + cx + e)}. \end{aligned}$$

Now we need this tangent to pass through (x_0, y_0) so

$$\begin{aligned} (y - y_0) &= \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(x_0,y_0)} (x - x_0) \\ (y - y_0) &= \frac{-(2ax_0 + cy_0 + d)}{(2by_0 + cx_0 + e)}(x - x_0) \\ (y - y_0)(2by_0 + cx_0 + e) &= (x_0 - x)(2ax_0 + cy_0 + d) \\ 2byy_0 + cx_0y + ey - 2by_0^2 - cx_0y_0 - ey_0 &= 2ax_0^2 + cx_0y_0 + dx_0 - 2axx_0 - 2cxy_0 - dx \\ 2axx_0 + 2byy_0 + c(x_0y + y_0x) + d(x + x_0) + e(y + y_0) &= 2ax_0^2 + 2by_0^2 + 2cx_0y_0 + 2dx_0 + 2ey_0 \\ 2axx_0 + 2byy_0 + c(x_0y + y_0x) + d(x + x_0) + e(y + y_0) &= -2f \\ axx_0 + byy_0 + c\frac{(x_0y + y_0x)}{2} + d\frac{(x + x_0)}{2} + e\frac{(y + y_0)}{2} + f &= 0 \end{aligned}$$

as required.

(b) You could make use of part (a) here to find the tangent at both of the points. However you could realise the significance of the chosen points. The tangent at $(1, 0)$ is just the vertical line $x = 1$, and the tangent at $(0, 1)$ is $y = 1$. These lines intersect at $(1, 1)$.

9. (a) Suppose this is not the case. Then draw the shortest line segment between the point P and the line L, call it m. Then drop the perpendicular from the point to the line. This is a right-angled triangle. So the hypotenuse (in this case m) is longer than the perpendicular. This is a contradiction and so the shortest line segment is indeed the perpendicular from the point to the line.

(b) Take an arbitary point (x, y) equidistant from the point and line. The square of the distance between this point and the line $x = -p$ is $(x + p)^2$ (it should make sense why we use the square of the distance rather than the distance itself in a moment). The square of the distance between this point and $(p,0)$ is $(x-p)^2+y^2$. So we need to solve

$$\begin{aligned}(x+p)^2 &= (x-p)^2+y^2\\ 4px &= y^2.\end{aligned}$$

This is the equation of a parabola with directrix $x=-p$ and focus $(p,0)$.

10. We can differentiate both sides to obtain

$$f'(x) = f(x)$$

and solve directly as follows:

$$\begin{aligned}\frac{f'(x)}{f(x)} &= 1\\ \ln|f(x)| &= x+c\\ \ln f(x) &= x+c\\ f(x) &= Ae^x.\end{aligned}$$

Using the condition $f(0)=1$ to find $A=1$. The range being the positive real numbers prevents constant functions from working.

11. $\int_0^1 x\,dx = \frac{1}{2}$ so we have

$$\int_0^1 f(x) - x\,dx = 0.$$

If $f(x)$ is identically x then the existence of x_0 is clear. Suppose this isn't the case, then there exists some x in the range such that $f(x) > x$. As the total integral is 0 it follows that there is some other x in range such that $f(x) < x$. By Intermediate Value Theorem, there must exist some x such that $f(x) = x$, which we take as our x_0.

12. The dot product gives $\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}||\mathbf{y}|\cos\theta$ and so $\mathbf{x}\cdot\mathbf{y} \le |\mathbf{x}||\mathbf{y}|$. But,

$$(\mathbf{x}\cdot\mathbf{y})^2 = \left(\sum_{i=1}^n x_i y_i\right)^2$$

and

$$(|\mathbf{x}||\mathbf{y}|)^2 = |\mathbf{x}|^2|\mathbf{y}|^2 = (\mathbf{x}\cdot\mathbf{x})(\mathbf{y}\cdot\mathbf{y}) = \sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2$$

and the desired inequality follows.

Equality holds when $\cos\theta = 1, \theta = 0$, i.e. when vectors $\mathbf{x},\mathbf{y}$ are parallel.

This is called the *Cauchy-Bunyakovski-Schwarz* (or, more commonly, the *Cauchy-Schwarz*) inequality.

13. The area of the parallelogram enclosed is given by $|\mathbf{a}||\mathbf{b}|\sin\theta$, so the square of the area is $|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta = |\mathbf{a}|^2|\mathbf{b}|^2(1-\cos^2\theta)$. But $\cos^2\theta = \frac{(\mathbf{a}\cdot\mathbf{b})^2}{|\mathbf{a}|^2|\mathbf{b}|^2}$. So the area is given by $|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 = 14\cdot 29 - 20^2 = 6$.

    Considering instead the cross-product we want the square of the modulus of $(-1,2,-1)^{\mathrm{T}}$, which is $1+4+1=6$.

    If you wish, you can demonstrate that this works in general for three-dimensional vectors. But that would require a lot of algebra.

14. (a) Expanding and simplifying shows that $(a-b)(c-d)+(a-d)(b-c)=(a-c)(b-d)$. So we can use the *triangle inequality* to deduce the desired result.

    (b) From a diagram it would be clear that we wish to find the absolute value of the difference in arguments between $a-b$ and $c-a$. So the angle is $|\arg\frac{b-a}{c-a}|$. (The absolute value is taken because the angles may be oriented in two different ways.)

    (c) Equality holds if there exist $m,n\in\mathbb{R}^+$ such that $m(a-b)(c-d)=n(a-d)(b-c)$, or equivalently that the ratio $\frac{(a-b)(c-d)}{(a-d)(b-c)}$ be a positive real number. This means that $|\arg\frac{(a-b)(c-d)}{(a-d)(b-c)}| = |\arg\frac{a-b}{a-d} + \arg\frac{c-d}{b-c}| = \pi$ RECHECK THIS.

15. First note that a polynomial has a Maclaurin series as the derivatives around zero are all defined. Let $\mathcal{P}_n(x) = \sum_{i=1}^n a_i x^i$ be a polynomial. It suffices to check the coefficient of x^i in the power series is also a_i.

    The ith derivative around zero of $\mathcal{P}_n(x)$ is the ith derivative of x^i evaluated at 0 and divided by $n!$. We only need consider the coefficient of x^i as higher order terms evaluated at zero disappear, and lower order terms also disappear during the differentiation process. The ith derivative of $a_i x^i$ is just $\frac{a_i i(i-1)(i-2)\ldots 1}{i!} = a_i$ so evaluated at zero this is still just the constant a_i.

16. As $t\to 0$ this inequality translates into the obvious $f(x_2)\le f(x_2)$. For this inequality to hold for all t in the given interval, it is necessary and sufficient the derivative on the RHS to be greater than or equal to the derivative on the LHS. Which translates to

    $$(x_1-x_2)f'(tx_1+(1-t)x_2) \le f(x_1)-f(x_2).$$

    Again as $t\to 0$ this reduces to

    $$(x_1-x_2)f'(x_2) \le f(x_1)-f(x_2)$$

    . Hmmm not sure.

17. Squaring and rearranging yields the inequality

$$(a_1 - a_2)^2 \geq 0$$

which clearly holds with equality if and only if $a_1 = a_2$.

We have

$$\begin{aligned}
\frac{\frac{a_1+a_2+\cdots+a_n}{n} + \frac{a_{n+1}+a_{n+2}+\cdots+a_{2n}}{n}}{2} &\geq \frac{\sqrt[n]{a_1 a_2 \ldots a_n} + \sqrt[n]{a_{n+1} \ldots a_{2n}}}{2} \\
&\geq \sqrt{\sqrt[n]{a_1 a_2 \ldots a_n} \cdot \sqrt[n]{a_{n+1} \ldots a_{2n}}} \\
&= \sqrt[2n]{a_1 a_2 \ldots a_{2n}}
\end{aligned}$$

as required.

(b) We wish to find a_{n+1} such that

$$\sqrt[n+1]{a_1 a_2 \ldots a_{n+1}} = \sqrt[n]{a_1 a_2 \ldots a_n}.$$

Raising both sides to power $n(n+1)$ and rearranging gives

$$a_{n+1} = \sqrt[n]{a_1 a_2 \ldots a_n}$$

as the value to choose for a_{n+1}.

(c) Using the choice for a_{n+1} found in the previous part, we are left to show

$$\frac{a_1 + a_2 + \cdots + a_n + \sqrt[n]{a_1 a_2 \ldots a_n}}{n+1} \geq \sqrt[n]{a_1 a_2 \ldots a_n}.$$

Rearranging this yields

$$\begin{aligned}
a_1 + a_2 + \cdots + a_n &\geq (n+1)\sqrt[n]{a_1 a_2 \ldots a_n} - \sqrt[n]{a_1 a_2 \ldots a_n} \\
a_1 + a_2 + \ldots a_n &\geq n\sqrt[n]{a_1 a_2 \ldots a_n} \\
\frac{a_1 + a_2 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \ldots a_n}
\end{aligned}$$

as required.

(d) Yes, they do. The claim being true for 2 variables means it is true for 4 and 8 and so on (all powers of 2). So raise to a high enough power and decrease by 1 at a time to obtain the inequality for a specific number of variables.

This is called *Cauchy Induction* and the inequality is called the *AM-GM Inequality*.

18. (a)

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \sin^n(x)\,dx \\
&= \int_0^{\frac{\pi}{2}} (1-\cos^2(x))\sin^{n-2}(x)\,dx \\
&= I_{n-2} - \int_0^{\frac{\pi}{2}} \cos x \cos x \sin^{n-1}(x)\,dx \\
&= I_{n-2} - \left[\frac{\sin^{n-1}(x)}{n-1}\cos x\right]_0^{\frac{\pi}{2}} - \frac{1}{n-1}\int_0^{\frac{\pi}{2}} \sin^n(x)\,dx \\
&= I_{n-2} - [0-0] - \frac{1}{n-1} I_n.
\end{aligned}$$

So rearranging gives $I_n = \frac{n-1}{n} I_{n-2}$.

19. (a) For small x, $\sin x \approx x$ so the limit is $\frac{x}{x} = 1$.

(b) Consider the function $\sin x$. It has roots at $x = 0, \pm\pi, \pm 2\pi, \ldots$, so it has roots at $x = 0, x^2 = \pi^2, 2^2\pi^2, \ldots$ (and only at these values). In the equation given, the RHS has roots at $\pi^2, 2^2\pi^2, \ldots$ and so we have by Factor Theorem

$$\sin x = Ax\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots$$

for some constant A. Dividing by x we have an equation similar to the given one, but with a constant left to find. To do this, we can take the limit as $x \to 0$. As found in part (a), the LHS approaches 1 (therefore so does the RHS). But the RHS approaches $A \cdot 1 \cdot 1 \cdots = A$ so $A = 1$ giving the result.

(c) Each term is of the form $\frac{4n^2}{(2n-1)(2n+1)} = \frac{4n^2}{4n^2-1}$. The sum converging to a non-zero value means its reciprocal also converges to a non-zero value, so in an effort to make this product resemble that found in part (b) we rewrite this as the reciprocal of $1 - \frac{1}{4n^2}$. Substituting $x = \frac{\pi}{2}$ into the result found in part (b) gives the result of the product as $\frac{\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \frac{\pi}{2}$. So the value of the infinite product is $\frac{\pi}{2}$.

20. (a) Solving the auxiliary equation $\lambda^2 + 4\lambda = 0$ we find $\lambda = \pm 2i$ and so the complementary solution is $y(x) = Ae^{2ix} + Be^{-2ix} = C\cos(2x) + D\sin(2x)$. Now considering the constant 2 we have the general solution $y(x) = C\cos(2x) + D\sin(2x) + \frac{1}{2}$. With initial conditions we find $C = -\frac{1}{2}$ and $2D = 0 \implies D = 0$ so the particular solution is $\frac{1}{2} - \frac{1}{2}\cos(2x) = \sin^2 x$.

(b) With initial conditions we have $y(0) = 0 = a_0$ and $y'(0) = 0 = a_1$. We have the identity (holds for every x)

$$\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} + 4\sum_{k=0}^{\infty} a_k x^k = 2.$$

For $k \geq 1$ the coefficient of x^k must be 0 from which the given recurrence relation follows. From here, we find that if $a_k = 0$, then $a_{k+2} = 0$ also. This means that $a_{2k+1} = 0$ for any natural k. To find the value of a_{2k} we need to find a_2 (as the recurrence is invalid with $k = 0$). Considering the constant terms we find $2 \cdot 1 \cdot a_2 + 4 \cdot a_0 = 2 \implies a_2 = 1$.

The expression for a_{2k} can be proved inductively (or in an inductive style - that is, writing a_{2k} in terms of a_{2k-2} and so on down to $a_2 = 2$).

Using this, we have

$$y(x) = \sum_{k=1}^{\infty} \frac{24^{k-1}(-1)^{k-1}x^{2k}}{(2k)!}.$$

With the initial conditions, this differential equation has a unique solution, and so this is the power series of $\sin^2 x$ (assuming it converges).

(c) We can check this power series by using the expansion of $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$. So we have

$$\begin{aligned}
\sin^2 x &= \frac{1}{2} - \frac{1}{2}\cos(2x) \\
&= \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} \\
&= -\frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} \\
&= \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1} 4^k (x)^{2k}}{(2k)!} \\
&= 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1} 4^{k-1} (x)^{2k}}{(2k)!} \\
&= \sum_{k=1}^{\infty} \frac{2(-1)^{k-1} 4^{k-1} (x)^{2k}}{(2k)!},
\end{aligned}$$

which is the series found in part (b).

21. (a)

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\, dx = \frac{1}{2}\int_0^L \left[\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right]\, dx.$$

Considering each of these separately, we have

$$\begin{aligned}
\int_0^L \cos\left(\frac{(m-n)\pi x}{L}\right)\, dx &= \frac{L}{(m-n)\pi}\left[\sin\left(\frac{(m-n)\pi x}{L}\right)\right]_0^L = 0 - 0 = 0 \\
\int_0^L \cos\left(\frac{(m+n)\pi x}{L}\right)\, dx &= \frac{L}{(m+n)\pi}\left[\sin\left(\frac{(m+n)\pi x}{L}\right)\right]_0^L = 0 - 0 = 0.
\end{aligned}$$

So the value of this integral is just zero whenever $m \neq n$. In the case that they are equal, the first integral would require dividing by zero and so its value would be incorrect. What we would have instead is

$$\int_0^L 1\,dx = L,$$

as $\cos 0 = 1$. So the value of the integral given is $\frac{L}{2}$ if $m = n$ and 0 otherwise.

(b) L is the period of this function.

$$\begin{aligned}
\frac{2}{L}\int_0^L [f(x)]^2\,dx &= \frac{2}{L}\int_0^L \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty}\left[b_m \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx\right] \\
&= \frac{2}{L}\sum_{n=1}^{\infty} b_n \cdot b_n \cdot \frac{L}{2} \\
&= \sum_{n=1}^{\infty} b_n^2
\end{aligned}$$

as required.

(c) Using the result in part(b), on the LHS, we have

$$\frac{2}{2}\int_0^2 x^2\,dx = \frac{8}{3}.$$

On the RHS, we have

$$\left(\frac{4}{\pi}\right)^2 \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

So we have

$$\begin{aligned}
\frac{8}{3} &= \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \\
\frac{\pi^2}{6} &= \sum_{n=1}^{\infty}\frac{1}{n^2}
\end{aligned}$$

as the required sum.

22. The answer is yes. Here we will prove the existence of such a matrix **A**. All arguments apply analogously to **B** so its existence would immediately follow.

We can prove first the 'only if' direction by noting that if $\mathbf{M}$ is non-singular, then we have

$$\begin{aligned} \mathbf{MA} &= \mathbf{O}_2 \\ \iff \mathbf{M}^{-1}\mathbf{MA} &= \mathbf{M}^{-1}\mathbf{O}_2 \\ \iff \mathbf{A} &= \mathbf{O}_2, \end{aligned}$$

but we need $\mathbf{A}$ to be non-zero and so the 'only if' part is proved.

The matrix $\mathbf{M}$ being singular is equivalent to $ad = bc$. Let

$$\mathbf{A} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

So we are solving

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Looking at each entry of the matrix on the LHS we might try setting $g = -c, e = d$ (to make use of the zero determinant, and to ensure that not all entries are 0). From here we can try setting $f = h = 0$ to find

$$\mathbf{MA} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & 0 \\ -c & 0 \end{pmatrix}$$

$$= \begin{pmatrix} ad-bc & 0 \\ dc-dc & 0 \end{pmatrix}.$$

With the condition that the determinant of $\mathbf{M}$ be zero the matrix $\mathbf{A}$ has been found, as required.

23. (a) The volume is given by

$$\pi \int_1^\infty \frac{1}{x^2}\,dx = -\pi \left[\frac{1}{x}\right]_1^\infty = \pi(0-(-1)) = \pi$$

as required.

(b) I is a diverging integral as its antiderivative ($\ln x$) diverges as $x \to \infty$. The surface area of the function $f(x) = \frac{1}{x}$, S, is given by

$$S = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx.$$

Notice that $\frac{1}{x^4} > 0$ for all positive x, so $\sqrt{1+\frac{1}{x^4}} > 1$ so we have

$$S > 2\pi \int_1^\infty \frac{1}{x}\,dx,$$

which we have established is divergent, so S itself is also divergent - Gabriel's horn has infinite surface area. *AT-J*

24. (a) i. We have that
$$f'(\alpha) = -\int_0^\infty \sin x\, e^{-\alpha x}\,\mathrm{d}x.$$
Using integration by parts,
$$f'(\alpha) = -\left(-\frac{\sin x\, e^{-\alpha x}}{\alpha}\bigg|_0^\infty + \frac{1}{\alpha}\int_0^\infty \cos x\, e^{-\alpha x}\,\mathrm{d}x\right).$$
Using integration by parts again,
$$f'(\alpha) = -\frac{1}{\alpha}\left(-\frac{\cos x\, e^{-\alpha x}}{\alpha}\bigg|_0^\infty - \frac{1}{\alpha}\int_0^\infty \sin x\, e^{-\alpha x}\,\mathrm{d}x\right).$$
So
$$f'(\alpha) = -\frac{1}{\alpha}\left(\frac{1}{\alpha} + \frac{1}{\alpha}f'(\alpha)\right)$$
therefore
$$\alpha^2 f'(\alpha) = -(1 + f'(\alpha))$$
and
$$(1+\alpha^2)f'(\alpha) + 1 = 0$$
hence
$$f'(\alpha) + \frac{1}{1+\alpha^2} = 0. \ \square$$

ii. By integrating, we have that
$$f(\alpha) = C - \tan^{-1}(\alpha)$$
for some constant C. Now,
$$\lim_{\alpha\to\infty} f(\alpha) = C - \frac{\pi}{2} = \lim_{\alpha\to\infty}\int_0^\infty \frac{\sin x}{x}e^{-\alpha x}\,\mathrm{d}x = 0$$
because the integral converges for all non-negative α, and by properties of even functions,
$$\int_{-\infty}^\infty \frac{\sin x}{x}\,\mathrm{d}x = 2\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = 2f(0) = \pi. \ \square$$

(b) i. We have that
$$g'(\alpha) = \int_0^\infty \frac{-x\sin(\alpha x)}{x^2+1}\,\mathrm{d}x = \int_0^\infty \frac{-x^2\sin(\alpha x)}{x(x^2+1)}\,\mathrm{d}x,$$
adding and subtracting $\sin(\alpha x)$ from the numerator,
$$g'(\alpha) = \int_0^\infty \frac{\sin(\alpha x)}{x(x^2+1)}\,\mathrm{d}x + \int_0^\infty \frac{-(x^2+1)\sin(\alpha x)}{x(x^2+1)}\,\mathrm{d}x$$
and so
$$g'(\alpha) = \int_0^\infty \frac{\sin(\alpha x)}{x(x^2+1)}\,\mathrm{d}x - \int_0^\infty \frac{\sin(\alpha x)}{\alpha x}\,\mathrm{d}(\alpha x)$$
therefore by (a) ii. we have that
$$g'(\alpha) + \frac{\pi}{2} = \int_0^\infty \frac{\sin \alpha x}{x(x^2+1)}\,\mathrm{d}x.$$
and on differentiating we see that

$$g''(\alpha) = \int_0^\infty \frac{x \cos \alpha x}{x(x^2+1)} \, \mathrm{d}x = g(\alpha). \; \square$$

ii. The auxiliary equation is $\lambda^2 = 1$ so the general solution to this second order ODE is

$$g(\alpha) = Ae^\alpha + Be^{-\alpha}$$

for some constants A, B. Now,

$$g(0) = A + B = \int_0^\infty \frac{1}{x^2+1} \, \mathrm{d}x = \frac{\pi}{2}$$

and

$$g'(0) = A - B = -\frac{\pi}{2},$$

therefore

$$g(\alpha) = \frac{\pi}{2} e^{-\alpha},$$

and by properties of even functions,

$$\int_{-\infty}^\infty \frac{\cos x}{x^2+1} \, \mathrm{d}x = 2 \int_0^\infty \frac{\cos x}{x^2+1} \, \mathrm{d}x = 2g(1) = \frac{\pi}{e}. \; \square$$

25. First integrate using the substitution $t = \tanh u$. So $(0, x) \to (0, \operatorname{artanh}(x))$ and $dt = \operatorname{sech}^2(u) \, du$. We have

$$\begin{aligned}
\int_0^x \frac{1}{1-t^2} \, dt &= \int_0^{\operatorname{artanh}(x)} \frac{\operatorname{sech}^2(u)}{1 - \tanh^2(u)} \, du \\
&= \int_0^{\operatorname{artanh}(x)} \frac{\operatorname{sech}^2(u)}{\operatorname{sech}^2(u)} \, du \\
&= \int_0^{\operatorname{artanh}(x)} 1 \, du \\
&= \operatorname{artanh}(x)
\end{aligned}$$

as $\operatorname{artanh}(0) = 0$.

Now integrate instead using partial fractions.

$$\begin{aligned}
\int_0^x \frac{1}{1-t^2} \, dt &= \int_0^x \frac{1}{(1-t)(1+t)} \, dt \\
&= \frac{1}{2} \int_0^x \frac{1}{1-t} + \frac{1}{1+t} \, dt \\
&= \frac{1}{2} \left[-\ln(1-t) + \ln(1+t) \right]_0^x \\
&= \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right).
\end{aligned}$$

It follows that this expression is equivalent to $\operatorname{artanh}(x)$. Note also the range of validity: $-1 < x < 1$.

26. (a) Note that $\cos x + \sin x = \sqrt{2}(\sin\left(x + \frac{\pi}{4}\right))$ (harmonic form). The coordinates of the maximum value on this curve are then clearly $\left(\frac{\pi}{4}, 2^{\frac{1}{4}}\right)$. The sketch of the curve is as follows: INSERT SKETCH OF CURVE.

(b) INSERT SKETCH.

(c) Here it may be easier to just work in the given form and use the formula for area enclosed in polar coordinates. Let R denote this area, then

$$\begin{aligned}\frac{1}{2}\int_{-\pi}^{\pi}\left(\sqrt{\cos(\theta)+\sin(\theta)}\right)^2 d\theta &= \frac{1}{2}\int_{-\pi}^{\pi}\cos(\theta)+\sin(\theta)\,d\theta \\ &= [\sin(\theta)]_0^{\pi} \\ &= CHANGELIMITS.\end{aligned}$$

3.2 Stats

1. (a) It is best to work with expectations, so that $\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$. Here we have

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathbb{E}((X+Y)^2) - [\mathbb{E}(X+Y)]^2 \\ &= \mathbb{E}(X^2) + 2\mathbb{E}(XY) + \mathbb{E}(Y^2) - [\mathbb{E}(X) + \mathbb{E}(Y)]^2 \\ &= \mathbb{E}(X^2) + 2\mathbb{E}(XY) + \mathbb{E}(Y^2) - [\mathbb{E}(X)]^2 - 2[\mathbb{E}(X)\mathbb{E}(Y)] - [\mathbb{E}(Y)]^2 \\ &= (\mathbb{E}(X^2) - [\mathbb{E}(X)]^2) + (\mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2) + 2(\mathbb{E}(XY) - [\mathbb{E}(X)\mathbb{E}(Y)]) \\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) + \mathrm{Cov}(X,Y)\end{aligned}$$

   (b) From here it is clear that a necessary and sufficient criterion is $\mathrm{Cov}(X,Y) = 0$.

   (c) Now assume that X and Y are independent. We want to show that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$. Starting with the LHS we have

$$\begin{aligned}\mathbb{E}(XY) &= \sum_{x_i}\sum_{y_i} x_i y_i \mathbb{P}(x_i, y_i) \\ &= \sum_{x_i} x_i \sum_{y_i} y_i \mathbb{P}(x_i, y_i) \\ &= \sum_{x_i} x_i \sum_{y_i} y_i \mathbb{P}(x_i)\mathbb{P}(y_i) \\ &= \sum_{x_i} x_i \mathbb{P}(x_i) \sum_{y_i} y_i \mathbb{P}(y_i) \\ &= \mathbb{E}(X)\mathbb{E}(Y).\end{aligned}$$

2. Each Bernoulli Trial B_i takes the value 1 with probability p and 0 with probability $q = 1-p$ so has expectation p and variance $p - p^2 = p(1-p) = pq$. A binomial variable with n trials has its distribution given by $B_1 + B_2 + \cdots + B_n$ and as these are independent variables we can sum the expectations and variances. For the expectation we sum n lots of p, so np and for variance in the same way we have npq.

3. (a) There are $\binom{n+m}{k}$ ways of choosing k beads from the pile. $\binom{n}{r} \cdot \binom{m}{k-r}$. Summing this over r gives $\sum_{r=0}^{k} \binom{n}{r} \cdot \binom{m}{k-r}$. This is all the ways of choosing k total objects from the pile, so is equal to $\binom{n+m}{k}$.

   (b)

$$\begin{aligned}\mathbb{P}(Z=z) &= \mathbb{P}(X+Y=z) \\ &= \sum_{x=0}^{z} \mathbb{P}(X=x, Y=z-x) \\ &= \sum_{x=0}^{z} \mathbb{P}(X=x) \cdot \mathbb{P}(Y=z-x) \\ &= \sum_{x=0}^{z} \binom{n}{x} p^x q^{n-x} \cdot \binom{m}{z-x} p^{z-x} q^{m-z+x} \\ &= \sum_{x=0}^{z} \binom{n}{x} \cdot \binom{m}{z-x} p^{x+z-x} q^{n-x+m-z+x} \\ &= p^z q^{n+m-z} \sum_{x=0}^{z} \binom{n}{x} \cdot \binom{m}{z-x}\end{aligned}$$

This sum is the same as in part (a), so we have

$$\mathbb{P}(Z = z) = p^z q^{n+m-z} \binom{n+m}{z}$$

Which we recognise as the binomial distribution with number of trials $n + m$ and probability of success p. That is, $Z \sim B(n + m, p)$.

4. Note: I didn't come up with this question at all, I only adjusted a few parts. Otherwise it was taken from
   (a) The distribution is geometric and the probability of getting heads on the kth throw is $\frac{1}{2^k}$.
   (b) The expected winnings are given by

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot 2}{2^k} = \frac{2}{3}$$

   (Geometric series).
   (c) Simply, the expected winnings are just the value associated to each outcome multiplied by the probability of that outcome. So the associated winnings with each k is $\frac{2^k}{k} \cdot \frac{1}{2^k}$.

$$\sum_{k=1}^{\infty} \left((-1)^{k-1} \frac{2^k}{k} \right) \frac{1}{2^k} = \sum_{k=1}^{\infty} \left((-1)^{k-1} \frac{1}{k} \right).$$

   So now using the series expansion for $\ln(1 + x) = x - \frac{x^2}{2} + \ldots$ with $x = 1$ shows the series we wish to evaluate is just $\ln(2)$.

   For the rearrangement part we can just note that $2k - 1, 2k, 2k + 1$ are the consecutive terms in the series. To show that the series value exceeds $\frac{5}{6}$ we have

$$\frac{1}{2k-1} + \frac{1}{2k+1} > \frac{2}{2k+1}$$

   So now we wish to show this expression is greater than $\frac{1}{2k}$

$$\begin{aligned} \frac{2}{2k+1} &> \frac{1}{2k} \\ \iff 4k &> 2k + 1 \\ \iff 2k &> 1 \\ \iff k &> \frac{1}{2} \end{aligned}$$

   which clearly holds so each set of three consecutive terms is positive (in our grouping). Looking only at the first term gives $1 + \frac{1}{3} - \frac{1}{2} = \frac{5}{6}$, which is clearly greater than 0.7, the upper bound given for $\ln(2)$ so the series converges to at least two distinct values.

   Note: this type of convergence is called *conditional* convergence. There's a lot about it online.

When a series converges in this way, a result of Riemann's says that it may be rearranged to converge to any real value.

5. We have

$$\mathbb{P}(X < b) = \mathbb{P}(Y < b)$$

immediately. We also have

$$\mathbb{P}(X < a) = \mathbb{P}(Y < a)$$

so

$$\begin{aligned} 1 - \mathbb{P}(X < a) &= 1 - \mathbb{P}(Y < b) \\ \mathbb{P}(X \geq a) &= \mathbb{P}(Y \geq a). \end{aligned}$$

But also, $\mathbb{P}(X \geq a) \cap \mathbb{P}(X < b) = \mathbb{P}(a \leq X < b)$ so the result follows.

6. Problem 6

7. As an off-spec note, you may have noticed that this is the Laplace transform of the function.

3.3 Mechanics

3.4 Extra Pure

1. (a) We need to show that the group axioms hold.

   Closure: if a and b are in the set of numbers, then $a \cdot b$ takes some value modulo p. Unless this value is 0, $a \cdot b$ is also in the original set. But it is not 0 as this would mean that p divides $a \cdot b$ but this is impossible as p is prime and a, b are both non-zero and not divisible by p.

   Associativity: the multiplication of numbers is associative.

   Identity: 1 is the multiplicative identity and it is in the given set.

   Inverse: For a given number a, we need some x such that

   $$ax \equiv 1 \pmod{p}.$$

   Equivalently, we need to be able to solve the equation

   $$ax + kp = 1$$

   for fixed integers a and p. But a and p are coprime and so using the result given at the start of the question, we know this equation has solutions.

   (b) We can choose any element in this group and generate a subgroup. The order of this subgroup is equal to the order of the element. By Lagrange's Theorem this order must divide p.

   (c)
   $$a^{p-1} \equiv a^{mk} \equiv (a^k)^m \equiv 1^m \equiv 1 \pmod{p}$$

   This question was about a proof of Fermat's Little Theorem using group theory. To see that it holds for any a we can consider the case of a being divisible by p separately (and the theorem is obviously true), and we can note that that set of integers modulo n partitions the set of integers.

2. (a) If n is composite, by definition there exist positive integers $a, b \neq 1$ such that $ab = n$. These numbers are in G. Take a. If it has an inverse c, then we have $ac \equiv 1 \pmod{n}$. So we have a solution to the equation $ac = kn + 1$ for some integer k. Let $n = an_1$ (existence of n_1 is clear as a divides n), then we have $a(c - kn_1) = 1$ where a divides the LHS but not the RHS, a contradiction. Therefore a has no inverse so this is not a group.

   (b) The 'only if' part of this claim is clear from part (a). Now suppose n is prime. $n - 1 \equiv -1 \pmod{n}$ so we need to show that $(n-2)! \equiv 1 \pmod{n}$. To see this, we can pair off inverses of each other. However, we need to also check which numbers are self-inverse as they cannot be paired off. That is, numbers x such that $x^2 \equiv 1 \pmod{n}$, or equivalently, $(x-1)(x+1) \equiv 0 \pmod{n}$, meaning $x = 1$ or $x = n - 1$. So we wish now to show that $(n-2)(n-3)\dots(2) \equiv 1 \pmod{n}$. But no elements here are self-inverse so each element can be paired with its inverse to obtain a product of '1's - the desired result follows. Note: this product relies on $n > 2$ as otherwise the product is of nothing. So the case of 2 should be checked separately - in which case the result clearly holds.

3. (a) The longest side must be shorter than the sum of the other two sides. That is, $a + 2d < a + a + d$, or $a > d$.

   (b) A is right-angled so we need to solve

$$a^2 + a^2 + 2ad + d^2 = a^2 + 4ad + 4d^2$$

   in integers, which amounts to

$$\begin{aligned} a^2 &= 2ad + 3d^2 \\ a^2 - 2ad + d^2 &= 4d^2 \\ (a-d)^2 &= 4d^2 \\ (a-d) &= \pm 2d \\ a &= d \pm 2d \\ a &= 3d. \end{aligned}$$

   (c) So the triangle lengths are $3r, 4r, 5r$. These are coprime if and only if $r = 1$.

4. (a) One of the side-lengths is a, so we must have a an integer. If $r \notin \mathbb{Q}$ then no integer multiple a of r is an integer so ar is not an integer. So we also have $r \in \mathbb{Q}$. We could have, for example, $a = 8, r = \frac{3}{2}$ meaning the triangle is *integer* but $r \notin \mathbb{Z}$.

   (b) Pythagoras' Theorem holds, that is,

$$\begin{aligned} a^2 + a^2r^2 &= a^2r^4 \\ 1 + r^2 &= r^4 \end{aligned}$$

   as required.

   (c) Let $r = \frac{p}{q}$, then we have

$$\begin{aligned} 1 + \frac{p^2}{q^2} &= \frac{p^4}{q^4} \\ q^3 + qp^2 &= \frac{p^4}{q}. \end{aligned}$$

   The LHS of this expression is integer, so it follows that the RHS is also integer. So q divides p^4, but q and p are coprime so it follows that $q = 1$. So any rational solution to this equation is also integer.

   (d) Solving the equation in integers, we want r such that

$$\begin{aligned} 1 + r^2 &= r^4 \\ 1 &= r^4 - r^2 \\ 1 &= r^2(r-1)(r+1). \end{aligned}$$

   If $r = 1$ then the RHS is 0. If $r > 1$ then the RHS is $\geq 4 \cdot 1 \cdot 3 = 12 > 1$ so there are no integer solutions to this equation. (And as such, there are no right-angled *integer* triangles.)

5. (a) Since $\mathbf{A} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is diagonal, the k-th power of $\mathbf{A}$ can be expressed as:

$$\mathbf{A}^k = \left(\mathbf{PDP}^{-1}\right)^k = \mathbf{PDP}^{-1}\mathbf{PDP}^{-1} \cdots \mathbf{PDP}^{-1} = \mathbf{PD}^k\mathbf{P}^{-1}$$

where $\mathbf{D}^k$ is obtained by raising each diagonal entry of $\mathbf{D}$ to the power k

(b) To find a matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$, we can define:

$$B = PD^{\frac{1}{2}}P^{-1}$$

where $\mathbf{D}^{\frac{1}{2}}$ is the diagonal matrix whose entries are the square roots of the corresponding entries in $\mathbf{D}$. Verifying:

$$\mathbf{B}^2 = (\mathbf{PD}^{\frac{1}{2}}\mathbf{P}^{-1})(\mathbf{PD}^{\frac{1}{2}}\mathbf{P}^{-1}) = \mathbf{PD}^{\frac{1}{2}}\mathbf{D}^{\frac{1}{2}}\mathbf{P}^{-1} = \mathbf{PDP}^{-1} = \mathbf{A}$$

(c) For $\mathbf{B}$ to be a real matrix, the eigenvalues of $\mathbf{A}$ (i.e. the diagonal entries of $\mathbf{D}$) must be non-negative. If any eigenvalue is negative, its square root would be non-real making $\mathbf{B}$ non-real.

(d) To compute $\mathbf{A}^{\frac{p}{q}}$ for integers p, q $(q > 0)$, we use:

$$A^{\frac{p}{q}} = \mathbf{PD}^{\frac{p}{q}}\mathbf{P}^{-1}$$

where $\mathbf{D}^{\frac{p}{q}}$ has entries $\lambda_i^{\frac{p}{q}}$, where λ_i are the eigenvalues of $\mathbf{A}$. The restrictions are:

- If q is even, all eigenvalues must be non-negative to ensure real results
- If q is odd, eigenvalues can be any real numbers (including negative).

A.G.

6. (a) f has a minimal value because it is necessarily positive (so bounded from below). To see that it does not have a maximal value, fix c and let a, b approach 0. Then the function approaches (from above) $\frac{c}{a+b}$ which is clearly unbounded.

$$\begin{aligned} f(a,b,c) &= \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \\ &= \frac{ta_1}{tb_1 + tc_1} + \frac{tb_1}{tc_1 + ta_1} + \frac{tc_1}{ta_1 + tb_1} \\ &= \frac{a_1}{b_1 + c_1} + \frac{b_1}{c_1 + a_1} + \frac{c_1}{a_1 + b_1} \\ &= f(a_1, b_1, c_1), \end{aligned}$$

so the result follows. The minimal value of $f(a,b,c)$ is attained for some numbers a, b, c but we have just demonstrated that this same value may be attained for some other 3 entries whose sum is 1 (as required).

(b) The given form of $f(a,b,c)$ follows from noting that $b + c = 1 - a$ (and analogously with the other fractions). $a, b, c > 0$ by condition and $a, b, c < 1$ as if any of the variables were greater than or equal to 1, the other two would need to be zero or negative, which cannot happen by condition.

(c)

$$\nabla f(a,b,c) = \left(\frac{1}{(1-a)^2}, \frac{1}{(1-b)^2}, \frac{1}{(1-c)^2}\right).$$

This is found by taking the partial derivatives of the function with respect to a then b then c. The actual differentiation can be carried out either by quotient rule, or by noting that

$$\frac{a}{1-a} = -\left(\frac{1-a+1}{1-a}\right) = -1 - \frac{1}{1-a}$$

and differentiating from there. Note that $\nabla g(a,b,c) = (1,1,1)$. By the result given at the start of the question (which may be used as we have found a valid constraint on the variables), we have

$$\begin{aligned}\frac{1}{(1-a)^2} &= \lambda \\ \frac{1}{(1-b)^2} &= \lambda \\ \frac{1}{(1-c)^2} &= \lambda \\ a+b+c &= 1.\end{aligned}$$

From here it follows that

$$\begin{aligned}\frac{1}{(1-a)^2} &= \frac{1}{(1-b)^2} \\ (1-b)^2 &= (1-a)^2 \\ (1-b-1+a)(1-b+1-a) &= 0 \\ (a-b)(2-(a+b)) &= 0.\end{aligned}$$

As a and b are strictly less than 1, it follows that $2 > (a+b)$ so the left factor in the equation is zero: $a = b$ as desired. In exactly the same way the equality of b and c follows.

So the minimal value of $f(a,b,c)$ is attained when $a = b = c = \frac{1}{3}$ (because of the constraint). The value of $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is $\frac{3}{2}$ and so the minimal value of $f(a,b,c)$ over all positive real values is $\frac{3}{2}$.

The inequality demonstrated is *Nesbitt's Inequality*. The result given at the start of the question is known as the *Lagrange Multipliers Theorem*. This result can also be used to demonstrate *Snell's Law* (from Physics), for example.

There is a small subtelty here to be investigated during first year maths courses. The question of whether or not f does actually achieve its extrema depends on the domain over which f is defined. The domain must be a *compact set*. But that is entirely off-spec.